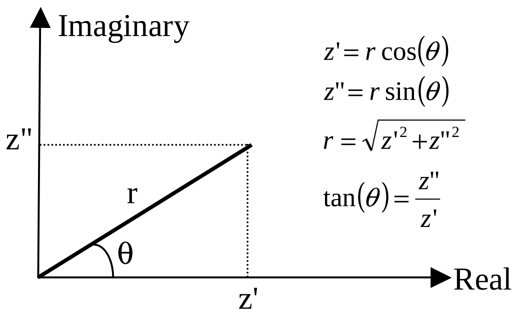
3.23 Fall 2006 – Recitation notes: Math overview

1. Complex Functions

General

Complex number: $z = z' + iz'' \Leftrightarrow z = r \cdot e^{i\theta}$



$$z' = r\cos(\theta)$$
$$z'' = r\sin(\theta)$$
$$r = \sqrt{z'^2 + z''^2}$$
$$\tan(\theta) = \frac{z''}{z'}$$

Function of a complex number: $f(z) = u(z', z'') + iv(z', z'')$

Differential of complex functions

How do we differentiate a complex function?

$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ ➔ depends on how Δz approaches o:

Example: $f(x + iy) = x + i2y$. What is the derivative at z=0?

$$\frac{df}{dz} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x + i2y}{x + iy} = \begin{cases} x \to 0,\ y \to 0 & \frac{i2y}{iy} = 2 \\ y \to 0,\ x \to 0 & \frac{x}{x} = 1 \end{cases}$$

⇒ A complex function is differentiable if *df/dz* exists and does not depend on the way *z*→*0*.

Cauchy-Riemann conditions:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y};\ \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \Rightarrow \text{f is differentiable}$$

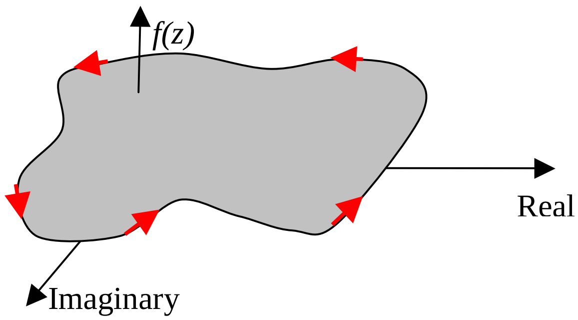
Some special functions

Trig. Functions: $e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow \cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right); \sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

Hyperbolic Functions: $\cosh(z) = \frac{1}{2}\left(e^{z} + e^{-z}\right); \sin(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right)$

Log function: $\ln(z) = \ln(r) + i\theta$

Integration of complex functions



2 options for integration: area integration and linear integration

Cauchy theorem & Residue theorem

Let's define a piece-wise regular closed curve in the complex plane as C.

If f(z) is analytic on a curve C, and within the area enclosed by C, then: $\oint_C f(z)dz = 0$

If f(z) has m isolated singularities within C, then $\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}\left(f(z_j)\right)$

Without going into detail let us introduce the "level" of a singularity:

If f(z) has a pole of order n at z_0 ⇔ $g(z) = f(z)(z - z_0)^n$ is analytic

$$\mathrm{Res}\left(f(z_0)\right) = \frac{1}{(n-1)!} \left. \frac{d^{n-1} g(z)}{dz^{n-1}} \right|_{z=z_0}$$

2. Linear Algebra

Vector Space

A linear vector space is a set of objects with a set of rules of manipulation.

1. Define a set S of vectors $\{u_i\}$
2. Define addition of two objects $\vec{u}_3 = \vec{u}_1 + \vec{u}_2$
    a. If u,v∈S ➔ u+v∈S
    b. There exists a zero vector **0**: u+**0**=u
    c. For every vector u∈S, there exsits an inverse v∈S, such that u+v=**0**
    d. Addition is commutative $u_1+u_2=u_2+u_1$
    e. Addition is associative $u_1+(u_2+u_3)=(u_1+u_2)+u_3$

3. Define multiplication by a number $\vec{u}_2 = c \cdot \vec{u}_1$
    a. If $u \in S$, $c \cdot u \in S$
    b. $\mathbf{1} \cdot u_1 = u_1$
    c. Multiplication is associative $c_1 \cdot (c_2 \cdot u_1) = (c_1 \cdot c_2) \cdot u_1$
    d. Distributive law of numbers $(c_1 + c_2) \cdot u_1 = c_1 \cdot u_1 + c_2 \cdot u_1$
    e. Distributive law of vectors $c_1 \cdot (u_1 + u_2) = c_1 \cdot u_1 + c_1 \cdot u_2$
4. Define a scalar product $u_1 \cdot u_2$=scalar
    a. $u_1 \cdot u_2$ is real
    b. $u_3 = c_1 u_1 + c_2 u_2$ ➔ $u_4 \cdot u_3 = c_1\, u_4 u_1 + c_2\, u_4 u_2$
    c. $u_1 u_1 \geq 0$ (equality for **0**)

NOTE:

1. We can find different ways of defining multiplication and addition, as well as different ways of defining the scalar product
2. Vector space requires addition and multiplication
3. Metric space is a linear space with a scalar product
4. vectors are usually represented as $\vec{u}$ or $|u\rangle$

Linear Operators

Can we define a function on a vector?

$c_1 = f(u_1)$ ➔ a function that returns a number. For example $\langle v|u\rangle = c$

$u_2 = f(u_1)$ ➔ a function that returns a vector. For example $\hat{A}|u\rangle = |v\rangle$

Linear Operators are defined over a linear vector space by demanding

Basis of a linear vector space

Since we said that combining two vectors results in another vector that is inside the linear space, one can pose the following question:

What is the minimum number of vectors needed in order to completely define the space?

In other words, can we find a set of vectors that we can use in order to create all the other vectors in the space?

We’ve seen many examples of that before:

- In the 3D vector space we needed the set of three vectors $\{\hat{x}, \hat{y}, \hat{z}\}$.
- When we talked about wavefunctions we used $\{u_n(x)\}$: $\psi(x) = \sum c_n u_n(x)$

In both cases the set of vectors is called the **basis** for the vector space, and the basis is said to **span** the space. The number of vectors required to form the basis are also referred to as the **dimension** of the vector space.

What we've also seen in class is that for a metric space (a space with an inner product) we can find the coefficients for the linear combination:

$$\vec{v} = \sum_n c_n \vec{u}_n \ ; c_n = \vec{v} \cdot \vec{u}_n$$

3. Fourier series

Sine and cosine as an orthogonal basis

Let's consider the set of functions sin(mx) and cos(nx), where m,n are integers. Let's consider the linear space which is the real, continuous, finite, and periodic functions in the {-π,π} range. The dot product on this space is given by: $< f \mid g >= \int_{-\pi}^{\pi} f(x)g(x)dx$. Let us consider the dot product between sine and cosine functions.

$$\int_{-\pi}^{\pi} Sin(mx)Cos(nx)dx = 0$$

$$Sin(mx)Cos(nx) = \frac{1}{2}\left(Sin(m-n)x + Sin(m+n)x\right)$$

$$\int Sin(mx)Cos(nx)dx = \frac{1}{2(m-n)}Cos(m-n)x + \frac{1}{2(m+n)}Cos(m+n)x$$

$$Cos(x) = Cos(-x) \Rightarrow \int_{-\pi}^{\pi} Sin(mx)Cos(nx)dx = 0$$

$$\int_{-\pi}^{\pi} Sin(mx)Sin(nx)dx = \pi \cdot \delta_{m,n}$$

$$Sin(mx)Sin(nx) = \frac{1}{2}\left(Cos(m-n)x - Cos(m+n)x\right)$$

$$\int Sin(mx)Sin(nx)dx = \frac{1}{2(m-n)}Sin(m-n)x - \frac{1}{2(m+n)}Sin(m+n)x$$

$$Sin(x) = -Sin(-x) \Rightarrow \int_{-\pi}^{\pi} Sin(mx)Sin(nx)dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} Cos(mx)Cos(nx)dx = \pi \cdot \delta_{m,n}$$

This means that all the sine and cosine functions are orthogonal to each other, and since there are an infinite number of them, they form a complete basis for all functions in that range. Turns out that in order for the basis to be complete, there's one more function needed: $u_0(x)$=const. You can check and see that u_0 is perpendicular to all the sine and cosine functions.

Instead of normalizing the functions Sine and Cosine, it is customary to re-define the inner product as $< f \mid g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$, which results in the sine and cosine being normalized. Under this normalization $u_0 = \sqrt{1/2}$.

Fourier series: part I

By the definitions of a linear space and a basis, any function in the range {-π,π} can be written as linear combination of sine and cosine functions:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n Cos(nx) + \sum_{n=1}^{\infty} b_n Sin(nx)$$

with: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Cos(nx)dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Sin(nx)dx$

(note that we've taken the $\sqrt{1/2}$ from a_0 and added it to u_0)

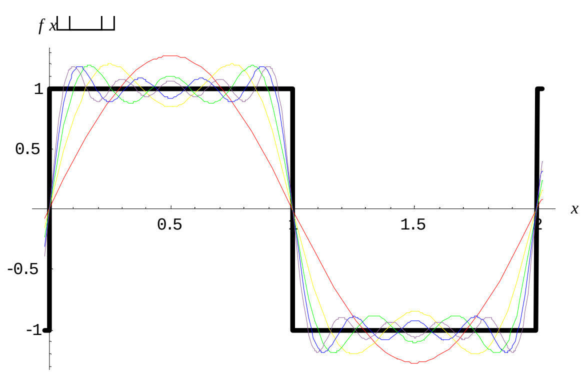
Dirichlet conditions & Gibbs phenomenon

Dirichlet conditions for a piecewise regular function f(x)

1. f(x) has a finite number of finite discontinuities and
2. f(x) has a finite number of extrema

Such a function can be expanded in a Fourier series $\Im$ which converges to the function at continuous points ($\Im(x) = f(x)$) and the mean of the positive and negative limits at points of discontinuity($\Im(x) = \frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\right)$).

At the discontinuity points one always observes what is known as the **Gibbs phenomenon**. The Fourier series (and other eigenfunction expansions) tend to oscillate strongly around the discontinuity, creating an "overshoot" near the edge.



The previous graph shows the expansion of a square wave with increasing numbers of terms. It illustrates both the convergence of $\Im$, and the Gibbs phenomenon.

Fourier series: part II

It can easily be shown that rather than looking at {-π,π} one can use any finite range $\{x_0, x_0+2L\}$ and rewrite the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n Cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n Sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{x_0}^{x_0+2L} f(x)dx \quad a_n = \frac{1}{L}\int_{x_0}^{x_0+2L} Cos(nx)dx \quad b_n = \frac{1}{L}\int_{x_0}^{x_0+2L} Sin(nx)dx$$

Furthermore, using Dirichlet conditions removes the requirement of having a periodic function.

⇒ For a periodic function $\Im(x)$ has the same period as the function and therefore can be used to represent f(x) for any x.

⇒ For a non-periodic function, we can use $\Im(x)$ to represent f(x) within a finite region in space.

Fourier series: part III

Another way of defining the Fourier series is using the exponential form, also known as the complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \text{ and } c_n = \frac{1}{2L}\int_{x_0}^{x_0+2L} f(x) e^{in\pi x/L} dx$$

One can show that the coefficients of the two forms are related:

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \\ \frac{1}{2}a_0 & n = 0 \end{cases}$$

Note that this is just a representation of the same functions in a different basis for the vector space.

Fourier Transforms

We saw that as long as the function is finite and continuous we can use $\Im(x)$ to represent f(x) (if f(x) obeys Dirichlet conditions then we can still use the Fourier series, but it will not be accurate at the discontinuity). Let us now expand the range 2L. As a mater of fact, let us expand the range in the special case of {-L/2,L/2}, where L→∞. Can we still use the Fourier series?

The answer is yes, as long as L is finite, but we run into trouble when L approaches infinity. Instead of discrete c_n numbers and an infinite sum we use an integral and a function c(n').

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i(nx/L)} \underset{\substack{n/L \to n' \\ c_n \to c(n')dn'}}{\Rightarrow} f(x) = \int_{-\infty}^{\infty} c(n') e^{2\pi i n' x} dn' \text{ and } c(n') = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n' x} dx$$

Putting things in the common notation *n'*→*k*, *c(n')*→*F(k)*, one can define the forward and inverse Fourier transforms:

$$F(k) = \Im(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx \Leftrightarrow f(x) = \Im^{-1}(F(k)) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k x} dk$$

In physics, you'd usually see a slightly modified form of the transforms:

$$F(k) = \Im(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \Leftrightarrow f(x) = \Im^{-1}(F(k)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dk$$

4. <u>Taylor series</u>

Taylor series are very useful because they provide a simplified expression for a given function that is a good numerical approximation around a certain value. This derives from the fact that any function can locally be approximated by a polynomial expression of arbitrary degree.

Here are a few examples:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots = \sum_{n=0}^{\infty} u^n$$, for u → 0

Now you can easily find the Taylor series for the natural logarithm:

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - \ldots)\,dt$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$, for x → 0

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$, for x → 0

Note: e^x must be equal to its derivative.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$, for x → 0

The cosine function simply follows from the derivative of the Taylor series of the sine function:

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x \\ &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$, for x → 0

5. Trigonometry

Some useful formulae:

$\tan(a) = \sin(a) / \cos(a)$

$\cos^2(a) + \sin^2(a) = 1$

$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$\sin^2(a/2) = (1 – \cos(a)) / 2$

$\cos^2(a/2) = (1 + \cos(a)) / 2$

6. Reference:

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